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Asymptotic expansions of logarithmic-exponential functions

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Abstract. The aim of this paper is to study the asymptotic expansion of real functions which are finite compositions of globally subanalytic maps with the exponential function and the logarithmic function. This is done thanks to a preparation theorem in the spirit of those that exist for analytic functions (Weierstrass) or subanalytic functions (Parusiński). The main consequence is that logarithmic-exponential functions admit convergent asymptotic expansion in the scale of real power functions. We also deduce a partial answer to a conjecture of van den Dries and Miller.

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Mathematical subject classification: 32B15, 32B20, 34E05.

1 Introduction

Before stating the definitions, let us briefly introduce and motivate this study.

The *globally subanalytic functions* are real functions whose graphs are globally subanalytic subsets of the Euclidian spaces $\mathbf{R}^n$, $n \in \mathbf{N}$. From the works of Lojasiewicz [Lo], Gabrielov [Ga] and Hironaka [Hi], we know that these functions have very nice geometric properties. Moreover, they can be reduced to some "normal forms" on suitable subanalytic cell decompositions. This result of Parusiński [Pa] (see also the preparation theorem of Lion and Rolin [LR1]) allows us to understand the asymptotic behaviour of such functions. If $f : (x, y) \mapsto f(x, y)$ is a globally subanalytic function of $\mathbf{R}^n \times \mathbf{R}$, then it admits finitely many reduced forms of the following type:

$$f(x, y) = |y - \theta(x)|^{\frac{p}{q}} A(x) V\left(c_1(x), ..., c_k(x), \frac{|y - \theta(x)|^{\frac{1}{q}}}{a(x)}, \frac{b(x)}{|y - \theta(x)|^{\frac{1}{q}}}\right)$$

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where the functions of $x \in \mathbf{R}^n$ are globally subanalytic, V is an analytic unit (i.e. an analytic function with values in a compact subset of $]0, +\infty[$) and p, q are integers. Hence we get an "expansion" of f in terms of functions such as $|y - \theta(x)|^r a(x)$ where $r \in \mathbf{Q}$ and θ, a are globally subanalytic.

In the case of one variable, this implies that globally subanalytic functions admit *convergent* asymptotic expansions in terms of $x^{p_n/q}$ where $p_n \in \mathbf{Z}$ and $p_n \to +\infty$.

A natural question is then: does there exist similar results for functions coming from more general problems of real analytic geometry such as polynomial or analytic differential equations? Such a question is very difficult even in the case of a polynomial vector field in the plane: it leads to the asymptotic study of the Poincaré return map and to Hilbert's 16th problem.

In order to simplify the problem, one can first consider elementary solutions of some Pfaffian equations in addition to the subanalytic functions. As illustrated in the two following examples, the real power functions and the exponential function appear naturally.

Example 1. Consider the Pfaffian 1-forms $\omega_\gamma = \gamma y dx - x dy$ with $\gamma \in \mathbf{R}_+ \setminus \mathbf{Q}$. The set $\{(x, x^\gamma) \mid x > 0\}$ is an integral curve of $\omega_\gamma = 0$ but is not subanalytic at the origin of $\mathbf{R}^2$.

Example 2. The set $\{(x, \exp(-1/x)) \mid x > 0\}$ is an integral curve of the Pfaffian equation $x^2 dy - y dx = 0$. But this is not a subanalytic set at the origin of $\mathbf{R}^2$.

This leads us to consider other classes of functions: the class of *x^λ-functions* and the class of *logarithmic-exponential functions*. The first one, introduced by Miller [Mi1] and Tougeron [To], contains the globally subanalytic functions and the real power functions but does not contain the exponential function. The second one contains the x^λ-functions and the functions exp and log. From the works of Miller [Mi1] in one hand, and the works of van den Dries, Macintyre and Marker [DMM1] in the other hand, we know that these functions are definable in some *o-minimal structures*. This property gives them an intrinsic geometric interest.

The x^λ-functions admit a preparation theorem [LR1] which is close to the preparation theorem for subanalytic functions. Hence we can easily derive asymptotic expansions of x^λ-functions of one variable. There also exists a preparation theorem for logarithmic-exponential functions. Unfortunately, this result of Lion and Rolin [LR1] does not give any precise asymptotic informations. Our aim in this paper is to give a preparation theorem for this class of functions which allows us to derive the *convergence* of some asymptotic expansions in certain scales of real functions such as the scale of real power functions.

2 Notations and results

We denote by $\mathbf{P}_1$ the real projective line with the standard analytic structure coming from the standard analytic structure of $\mathbf{R}$. We will consider $\mathbf{R}^n$ as a subset of $\mathbf{P}_1^n$ embedded in $\mathbf{P}_1^n$ and consequently, the subsets of $\mathbf{R}^n$ will be considered as subsets of the real n-dimensional torus. We suppose that the functions such as log or power functions are defined on $\mathbf{R}$ and equal 0 out of $]0, +\infty[$. A real power map $\Gamma : \mathbf{R}^p \to \mathbf{R}^p$ is the data of p real numbers $(\gamma_1, ..., \gamma_p)$ and is defined naturally by $\Gamma(x_1, ..., x_p) = (x_1^{\gamma_1}, ..., x_p^{\gamma_p})$. An *elementary* log-exp-*map* is a map $f = (f_1, ..., f_p) : \mathbf{R}^n \to \mathbf{R}^p$ such that each coordinate f_i is a coordinate x_j or the logarithm of a coordinate or the exponential of a coordinate.

A subset X of $\mathbf{R}^n$ is a *globally semianalytic set* if it is defined, in a neighbourhood of any point of $\mathbf{P}_1^n$, by a finite number of equalities and inequalities satisfied by analytic functions. A *globally subanalytic set* of $\mathbf{R}^n$ is the image of a globally semianalytic set of $\mathbf{R}^n \times \mathbf{R}^m$ by the canonical projection from $\mathbf{R}^n \times \mathbf{R}^m$ to $\mathbf{R}^n$.

A function $f : \mathbf{R}^n \to \mathbf{R}$ is said to be a *globally subanalytic function* if its graph is a globally subanalytic subset of $\mathbf{R}^n \times \mathbf{R}$. We will note $\mathfrak{S}_n$ the collection of all such functions of $\mathbf{R}^n$ and $\mathfrak{S} = \bigcup_{n\geq 0} \mathfrak{S}_n$. A map $f : \mathbf{R}^n \to \mathbf{R}^p$ is a *globally subanalytic map* if its coordinate functions are globally subanalytic functions.

Let us recall the definition of an *o-minimal structure* (see [Dr] and [DM] for more informations).

Definition 2.1. *Let $\mathfrak{A}_n$ be a collection of subsets of $\mathbf{R}^n$ and $\mathfrak{A} = \bigcup_{n\in\mathbf{N}} \mathfrak{A}_n$. We say that $\mathfrak{A}$ is an o-minimal structure if:*

1. *$\mathfrak{A}_n$ is a boolean subalgebra of $\mathbf{P}(\mathbf{R}^n)$ for all $n \in \mathbf{N}$.*
2. *The real semialgebraic subsets of the spaces $\mathbf{R}^n$, $n \in \mathbf{N}$, belong to $\mathfrak{A}$.*
3. *The elements of $\mathfrak{A}$ are stable under linear projection and cartesian product.*
4. *A subset of $\mathbf{R}$ which belongs to $\mathfrak{A}$ is a finite union of points and intervals.*

If $\mathfrak{A}$ is an o-minimal structure, a subset in $\mathfrak{A}$ is called a $\mathfrak{A}$-*set* or a set *definable in* $\mathfrak{A}$. A $\mathfrak{A}$-map from $\mathbf{R}^n$ to $\mathbf{R}^p$ is a map whose graph is definable in $\mathfrak{A}$.

To each o-minimal structure $\mathfrak{A}$ corresponds the class of $\mathfrak{A}$-functions. Such a class is stable under the elementary algebraic operation $+$ and $\cdot$, under composition and contains the semialgebraic functions. Conversely, given a class of real functions which is stable under $+$, $\cdot$ and the composition, which contains the semialgebraic functions, we can define a structure by taking the graphs of all the functions of the class. If this structure is o-minimal then we say that this class of functions is also *o-minimal*. In the sequel, we always deal with o-minimal structures or o-minimal classes of functions.

From Gabrielov's Theorem [Ga], the globally subanalytic sets form an o-minimal structure. When we add the graphs of the functions $x \mapsto x^\lambda$, $\lambda > 0$, or the graph of the exponential function, we still obtain o-minimal structures. Let us first define the class of x^λ-functions and of logarithmic-exponential functions introduced in the preceding paragraph.

Definition 2.2. *A* x^λ-map $f : \mathbf{R}^n \to \mathbf{R}^p$ *is finite composition of globally subanalytic maps and power maps. A* x^λ-function *is a* x^λ*-map from* $\mathbf{R}^n$ *to* $\mathbf{R}$.

Definition 2.3. *A* $\mathfrak{LE}$-map $f : \mathbf{R}^n \to \mathbf{R}^p$ *is a finite composition of globally subanalytic maps with elementary* log-exp-*maps. A* $\mathfrak{LE}$-function *is a* $\mathfrak{LE}$*-map from* $\mathbf{R}^n$ *to* $\mathbf{R}$.

The result of Miller can then be stated as follows.

Theorem 2.1 ([Mi1]). *The class of* x^λ*-functions is o-minimal. The corresponding o-minimal structure is* $\mathfrak{S}(\mathbf{R}_{an}^{\mathbf{R}})$.

The o-minimality of the class of $\mathfrak{LE}$-functions is due to van den Dries, Macintyre and Marker (following the works of Wilkie [Wi]).

Theorem 2.2 ([DMM1]). *The class of* $\mathfrak{LE}$*-functions is o-minimal. The corresponding o-minimal structure is* $\mathfrak{S}(\mathbf{R}_{an,\exp})$.

As we are interested in the asymptotic behaviour of functions definable in some o-minimal structure, we introduce the following definition.

Definition 2.4. *Let* $\mathfrak{A}$ *be a collection of functions on* $\mathbf{R}^n$, $n \in \mathbf{N}$. *A function* g *is said to be* $\mathfrak{A}$-comparable *with the functions* $g_1, \dots, g_p$ *on a set* E *if* g *has no root on* E *and if there exists a* $\mathfrak{A}$*-function* G *such that* $|G(g_1, \dots, g_p)/g|$ *has values in a compact subset of* $]0, +\infty[$.

In this paper, most of the asymptotic studies will be made at the origin. Thus we define the iterates of the function log as follows.

Definition 2.5. *We put* $\ell_0(x) = x$. *We define by induction the function* ℓ_i, $i > 0$, *in the following way: for all* x, $\ell_i(x) = \log|\ell_{i-1}(x)|$. *We say that* ℓ_i *is the* i*-th iterate of the function* log.

We now define two scales of real functions:

Rational scale. Let $\mathfrak{E}_{\mathbf{Q}}$ be the collection of functions of the form $\prod_{i=0}^{\infty} |\ell_i(x)|^{q_i}$, where the q_i's are rational and equal to 0 except for a finite number of indices i.

Real scale. Let $\mathfrak{E}_{\mathbf{R}}$ be the collection of functions of the form $\prod_{i=0}^{\infty} |\ell_i(x)|^{\alpha_i}$, where the α_i's are real and equal to 0 except for a finite number of indices i.

In [LR1] the class of $\mathfrak{L}\mathfrak{A}$-*functions* is introduced: in the case of one variable, they are finite compositions of globally subanalytic maps and the function log. There exists a preparation theorem for such functions [LR1] which implies that they admit convergent asymptotic expansions in the scale $\mathfrak{E}_{\mathbf{Q}}$. As we deal with o-minimal structures containing $\mathfrak{S}(\mathbf{R}_{an}^{\mathbf{R}})$, we first introduce another class of functions admitting convergent asymptotic expansions in the scale $\mathfrak{E}_{\mathbf{R}}$.

Definition 2.6. *Let f be a function from $\mathbf{R}^n \times \mathbf{R}$ to $\mathbf{R}$.*

1. *The function f is a $\mathfrak{L}\mathfrak{X}$-*function *of type* 0 *in the variable y if $f(x, y) = F(a(x), y)$ where F is a x^λ-function and $a = (a_1, ..., a_m)$ is a $\mathfrak{L}\mathfrak{E}$-map of $\mathbf{R}^n$.*

2. *The function f is a $\mathfrak{L}\mathfrak{X}$-function of type r in the variable y if*

$$f(x, y) = F(f_1(x, y), ..., f_m(x, y), \log f_{m+1}(x, y), ..., \log f_{m+\ell}(x, y))$$

where F is a x^λ-function and the f_i's are $\mathfrak{L}\mathfrak{X}$-functions of type less or equal to $r - 1$ in y.

Following the ideas of [LR1], we can prove a preparation theorem for the class of $\mathfrak{L}\mathfrak{X}$-functions. It is based on a preparation theorem for the class of x^λ-functions. This is the following result.

Theorem 2.3. *Let f be a $\mathfrak{L}\mathfrak{X}$-function of $\mathbf{R}^{n+1}$. There exists a finite partition of $\mathbf{R}^n \times \mathbf{R}$ into $\mathfrak{L}\mathfrak{E}$-cylinders such that, on each cylinder C, f admits the following expression:*

$$f_{|C}(x, y) = y_0^{\alpha_0} ... y_r^{\alpha_r} A(x) V(\varphi_1(x), ..., \varphi_k(x), m_1(x, y), ..., m_\ell(x, y))$$

where $y_0 = |y - \theta_0(x)|$, $y_1 = |\log y_0 - \theta_1(x)|$,..., $y_r = |\log y_{r-1} - \theta_r(x)|$, the functions θ_i being $\mathfrak{L}\mathfrak{E}$-functions of $\mathbf{R}^n$ identically equal to 0 *or comparable with y_{i-1} for all i. The function A is a $\mathfrak{L}\mathfrak{E}$-function, the functions φ_j are $\mathfrak{L}\mathfrak{E}$-fonctions with values in $[-1, 1]$, the m_j's have values in $[-1, 1]$ and are of the form*

$$m_j(x, y) = y_0^{\alpha_0^j} ... y_r^{\alpha_r^j} a_j(x)$$

where the a_j's are $\mathfrak{L}\mathfrak{E}$-functions. The exponents α_i and α_i^j are real. At last, V is an analytic unit in a neighbourhood of $[-1, 1]^{k+\ell}$. The function f is said to be reduced *in the system of variables $(y_0, ..., y_r)$.*

As the proof of Theorem 2.3 is very close to the proof of the preparation theorem for $\mathfrak{L}\mathfrak{A}$-functions [LR1], we do not give the details here. The following corollary will be used in the sequel.

Corollary 2.1. *Let f be a $\mathfrak{L}\mathfrak{X}$-function on $]0,1[$. Then f admits a convergent asymptotic expansion at the origin in the scale $\mathfrak{E}_{\mathbf{R}}$. Moreover, the monomials satisfy:*

1. *There exists $N \in \mathbf{N}$ such that, for every $m = \prod_{i=0}^{\infty} |\ell_i(x)|^{\alpha_i}$ of the expansion of f, $\alpha_i = 0$ if $i > N$.*
2. *The exponents of x in the monomials can accumulate only at $+\infty$.*

The main result of this paper is the following preparation theorem for $\mathfrak{L}\mathfrak{E}$-functions of one variable. It was annouced in [So].

Theorem 2.4. *Let $f : \mathbf{R} \to \mathbf{R}$ be a $\mathfrak{L}\mathfrak{E}$-function. There exists a partition of $\mathbf{R}$ into finitely many intervals I_n such that, on I_n, f admits the following expression:*

$$f(x) = z(x).\prod_{i=1}^{s} y_i(x)^{\delta_i}.V(z_1(x), ..., z_p(x), m_1(x), ..., m_q(x))$$

where

1. *$y_i = \exp h_i$, h_i being a $\mathfrak{L}\mathfrak{E}$-function of one variable for all i and $\delta_i \in \mathbf{R}$.*
2. *The function z is a $\mathfrak{L}\mathfrak{X}$-function.*
3. *$m_k(x) = y_1^{\alpha_1^k}...y_s^{\alpha_s^k} a_k(x)$, a_k being a $\mathfrak{L}\mathfrak{X}$-function for all k and $(\alpha_1^k, ..., \alpha_s^k) \in \mathbf{R}^s \setminus \{0\}$. The functions z_j are $\mathfrak{L}\mathfrak{X}$-functions for all j. Moreover, the functions m_k and z_j have values in $[-1, 1]$.*
4. *For all i, for all finite family $(g_1, ..., g_h)$ of $\mathfrak{L}\mathfrak{X}$-functions, the function y_i is not x^λ-comparable with the functions $y_1, ..., y_{i-1}, y_{i+1}, ..., y_s, g_1, ..., g_h$.*
5. *V is an analytic unit in a neighbourhood of $[-1, 1]^{p+q}$.*

Remark. The condition 4 in the statement of the theorem implies that the functions y_i admit very different asymptotic behaviours on each interval I of the partition. In particular, a function

$$m_k(x) = y_1^{\alpha_1^k}...y_s^{\alpha_s^k} a_k(x)$$

can not be comparable with a real power of x on I.

The main practicle consequence of Theorem 2.4 is the following result.

Theorem 2.5. *Let* $f :]0, \varepsilon[\to \mathbf{R}$ *be a* $\mathfrak{LE}$*-function. Assume* f *admits a formal asymptotic expansion in the scale* $\mathfrak{E}_{\mathbf{R}}$*. Then this expansion is convergent.*

Some results in the same spirit appear in [DMM2] with proofs based on Model Theory. Here, we only use arguments from analytic geometry. As a direct application of Theorem 2.5, we can prove the following result on the integrals of x^λ-functions.

Proposition 2.1. *There exist a* x^λ*-function* $f :]0, 1[^2 \to \mathbf{R}$ *and two* x^λ*-functions* $\varphi, \psi :]0, 1[\to [0, 1]$ *such that the function*

$$F(x) = \int_{\varphi(x)}^{\psi(x)} f(x, y)dy$$

does not belong to the class of $\mathfrak{LE}$*-functions.*

This negative result completes the study of the integrals of x^λ-functions made in [So]. In particular, it shows that there are strong differencies between the integrals of globally subanalytic functions and x^λ-functions. Indeed, as it is shown in [LR2], the integration of a globally subanalytic function on the fibers of a globally subanalytic function leads to a real function definable in the o-minimal class of $\mathfrak{LE}$-functions. Proposition 2.1 shows that real ramifications of the globally subanalytic functions imply a great change of behaviour under integration.

The paper is organized as follows. In the next section, we give the proof of Theorem 2.4. In section 4, we derive two consequences: Theorem 2.5 and Proposition 2.1. As an other application, the last section is devoted to a partial answer to a conjecture of van den Dries and Miller concerning o-minimal structures lying between $\mathfrak{S}(\mathbf{R}_{an}^{\mathbf{R}})$ and $\mathfrak{S}(\mathbf{R}_{an,\exp})$.

3 Preparation Theorem for $\mathfrak{LE}$-functions of one variable

The proof of Theorem 2.4 is based on the *combinatorial* idea of the proof of Proposition 3 in [LR1]. Indeed, we proceed by induction on the exponential complexity of the function f and we add the condition of non-comparability. Let us first recall some definitions.

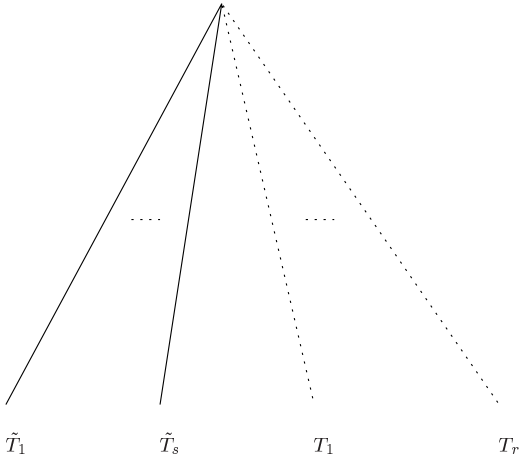
The tree of a $\mathfrak{LE}$-function on an interval. Let f be $\mathfrak{LE}$-function of $\mathbf{R}$. It is a finite composition of x^λ-maps with the functions exp and log. To all expression of f on an interval, we associate a *tree* which describes the way the functions log and exp are composed.

a. If f is a $\mathfrak{L}\mathfrak{X}$-function, we associate to it a vertex.

b. If f has the following form:

$$f = F(\exp h_1, ..., \exp h_s, \log g_1, ..., \log g_r, z_1, ..., z_t)$$

where the h_i's and the g_j's are $\mathfrak{L}\mathfrak{E}$-functions, the z_k's are $\mathfrak{L}\mathfrak{X}$-functions and F is a x^λ-function, we note T_j the tree associated to g_j and $\tilde{T}_i$ the tree associated to h_i. Then we associate to f the following tree:



The full edges correspond to the exponentials and the dotted edges correspond to the logarithms. This way, we get a tree whose vertices correspond to the $\mathfrak{L}\mathfrak{X}$-functions which appear in the expression of f, the dotted edges to the function log and the full edges to the function exp.

A tree (corresponding to an expression of f on an interval) is *simple* if its full egdes form a tree with the same *root* as the entire tree. We will also say that f admits a *simple expression*.

Now, we associate three integers to the expression of f on an interval:

1. If f is a $\mathfrak{L}\mathfrak{X}$-function, we say that its *exponential chain* is empty. If f has the form $f = F(\exp h_1, ..., \exp h_s, \log g_1, ..., \log g_r, z_1, ..., z_t)$, its exponential chain is the union of the set $\{\exp h_1, ..., \exp h_s\}$ with the exponential chains of the h_i's and the g_j's. The *exponential number* $e(f)$ is the cardinal of the exponential chain of f.

2. The *depth* of a vertex is the length of the edge which joins the root of the tree and the vertex. A *maximal vertex* is a vertex of maximal depth among those whose ascending edge is dotted and at least one of the descending edge is full. The number $h(f)$ is equal to the depth of the maximal vertices or 0 if there are no such vertices.

3. The number $c(f)$ is equal to the total number of full edges which descend from the maximal vertices or 0 if there are no such vertices.

Let us now turn to the proof of the theorem.

Proof. We split the proof in two steps. In the first one, we show that we can always find a partition of $\mathbf{R}$ into finitely many intervals such that f admits a simple expression on each interval. In the second step, we show how to get the reduced form of f using the preparation theorem for x^{λ}-functions.

In the sequel, we will use several times the following result [Wi],[DMM1]: if C is a x^{λ}-subset of $\mathbf{R}^N$ and if g is a $\mathfrak{LE}$-map from $\mathbf{R}$ to $\mathbf{R}^N$ then the inverse image of C by g, $g^{-1}(C)$, is a finite union of points and intervals.

Step 1. We proceed by induction on the 3-tuples (e, h, c) ordered with the lexicographic order.

- If $e = 0$ then f is a $\mathfrak{LX}$-function and its tree is clearly simple. If $h = 0$ then $c = 0$ and the tree of f is also simple.
- Let I be an interval on which f admits an expression given by the 3-tuple (e, h, c). We can assume that $c > 0$. Let us consider a maximal vertex V of the tree associated to f. It corresponds to a *sub-expression* of f of the form:

$$t(x) = \log T(x) = \log F(y_1(x), ..., y_s(x), z_1(x), ..., z_r(x))$$

where $F(v_1, ..., v_s, u_1, ..., u_r)$ is a x^{λ}-function of $\mathbf{R}^{s+r}$, the z_j's are $\mathfrak{LX}$-functions, $y_i = \exp h_i$ and the trees associated to the h_i's are simple.

We can always assume that $\exp h_1$ is not a sub-expression of any $\exp h_k$, $k > 1$. Let us prepare the function F with respect to the variable v_1 (see [Pa] or [LR1]). The space $\mathbf{R}^{s+r}$ splits into finitely many x^{λ}-cylinders on which F is reduced. Let $\tilde{C}$ be one of these cylinders. From the remark of the beginning of the proof, the $\mathfrak{LE}$-subset

$$D = \{x \in I \mid (y_1(x), ..., y_s(x), z_1(x), ..., z_r(x)) \in \tilde{C}\}$$

is a finite union of intervals (sub-intervals of I). Let J be one these intervals. There are two possibilities for the x^{λ}-cylinder C:

a. There exists two positive constants $0 < k < K$ and a x^λ-function θ such that the following inequalities hold on C:

$$k\theta(v_2, ..., v_s, u_1, ..., u_r) \leq v_1 \leq K\theta(v_2, ..., v_s, u_1, ..., u_r).$$

This means that v_1 is comparable with θ on C. It implies that y_1 must be comparable with the function $\theta(y_2, ..., y_s, z_1, ..., z_r)$ on the interval J. Then it comes:

$$\exp h_1 = \theta . \exp(h_1 - \log|\theta|)$$

where the function $h_1 - \log|\theta|$ has values in a compact subset of $\mathbf{R}$. We can then replace each expression $\exp h_1$ by the function $\theta . \exp(h_1 - \log|\theta|)$ in the expression of f. As the exponential function is analytic in a neighbourhood of the closure of the image of $h_1 - \log|\theta|$, we get a new expression of f with an exponential number strictly less than e on J. We can then use the hypothesis of induction to conclude.

b. The function F admits a reduction on C without translation term. It comes:

$$F(y_1, ..., y_s, z_1, ..., z_r) = y_1^{\alpha_1} A(y_2, ..., y_s, z_1, ..., z_r) U$$

where A is a x^λ-function and U is a x^λ-unit. The expression of t becomes:

$$t = \alpha_1 h_1 + \log A(y_2, ..., y_s, z_1, ..., z_r) + \log U.$$

As the function U is a x^λ-unit, we can assume that it has values in $[1/2, 2]$.Hence $\log U$ is a x^λ-function in the variables $y_1, ..., y_s, z_1, ..., z_r$. Consequently, we have:

- If $c > 1$ then we describes f with a new expression $\tilde{f}$ such that $c(\tilde{f}) < c$.
- If $c = 1$ then we describes f with a new expression $\tilde{f}$ such that $h(\tilde{f}) < h$.

In the two preceding cases, we use the hypothesis of induction to conclude. We have then proved that, up to a finite decomposition of I into intervals, we can always get a simple expression of f. This ends the first step.

Step 2. From the first step, we can assume that there exists a partition of $\mathbf{R}$ into finitely many intervals I on which f can be written:

$$f = F(y_1, ..., y_s, z_1, ..., z_r) \tag{*}$$

where F is a x^λ-function, $y_i = \exp h_i$, the h_i's being $\mathfrak{L}\mathfrak{E}$-functions and the z_j's being $\mathfrak{L}\mathfrak{X}$-functions. The 3-tuple of f is then of the form $(e, 0, 0)$ and we have $0 \leq s \leq e$. We proceed now by induction on e. Let $\mathbf{P}_e$ be the following proposition:

$\mathbf{P}_e$: *Let* $f_n = F_n(y_1, ..., y_s, z_1, ..., z_r)$ *be a finite family of* $\mathfrak{L}\mathfrak{E}$*-functions with exponential numbers less or equal to* e *on* I*, the* F_n*'s being* x^λ*-functions for all* n*. There exists a finite partition of* I *into sub-intervals (possibly reduced to a point) on which the* f_n*'s are reduced as stated in the theorem.*

The case $e = 0$**.** The f_n's are $\mathfrak{L}\mathfrak{X}$-functions and it suffices to apply Theorem 2.3.

Step of induction. We assume that $e > 0$ and we proceed by induction on the integer $s \leq e$.

$s = 1$. We have $f_n = F_n(y_1, z_1, ..., z_r)$ where $F_n(v_1, u_1, ..., u_r)$ is a x^λ-function of $\mathbf{R}^{1+r}$ for all n. Let us simultaneously prepare the F_n's with respect to the variable v_1. We get a finite partition of $\mathbf{R}^{1+r}$ into x^λ-cylinders on which the F_n's are reduced. Let $\tilde{C}$ be one of these cylinders. Its inverse image by the map $x \mapsto (y_1(x), z_1(x), ..., z_r(x))$ is a finite union of intervals. Let J be one of these intervals.

There are two cases:

- **a.** There exists a finite family of $\mathfrak{L}\mathfrak{X}$-functions $(w_1, ..., w_p)$ and a x^λ-function θ such that y_1 is comparable with $\theta(w_1, ..., w_p)$ on J. As in the case **a** of the first step, we deduce that the f_n's have exponential numbers strictly less than e on J. Hence we apply the hypothesis of induction $\mathbf{P}_{e-1}$ to conclude.

- **b.** There is no such x^λ-comparability relation. Thus the function F_n are reduced to the form:

$$F_n(y_1, z_1, ..., z_r) = y_1^{\alpha_0^n} A_n(z_1, ..., z_r) U_n.$$

  This gives the required expression for the functions f_n.

Let us pass to the step of induction on the integer s. We note $\bar{y} = (y_2, ..., y_s)$ and $z = (z_1, ..., z_r)$ in such a way that $f_n = F_n(y_1, \bar{y}, z)$. We prepare the F_n's with respect to the variable v_1. We get a finite partition of $\mathbf{R}^{s+r}$ into x^λ-cylinders $\tilde{C}$ to which corresponds a finite partition of I into sub-intervals J. We argue as in the case $s = 1$. Each time we get a x^λ-comparability relation on J between y_1 and some other variables $(\bar{y}, w)$, w being a $\mathfrak{L}\mathfrak{X}$-map, we can apply the hypothesis of induction $\mathbf{P}_{e-1}$. Consequently, we can assume that there is no such relation of comparability. It comes:

$$F_n(y_1, \bar{y}, z) = y_1^{\mu_0^n} A_n(\bar{y}, z) U_n((\bar{y}, z), y_1^{\mu_1}, ..., y_1^{\mu_\ell})$$

where the A_n's are x^λ-functions and the U_n's are x^λ-units of the form

$$V_n\left(c_1(\bar{y},z),\ldots,c_m(\bar{y},z),\frac{y_1^{\mu_1}}{a_1(\bar{y},z)},\ldots,\frac{y_1^{\mu_\ell}}{a_\ell(\bar{y},z)},\frac{b_1(\bar{y},z)}{y_1^{\mu_1}},\ldots,\frac{b_\ell(\bar{y},z)}{y_1^{\mu_\ell}}\right).$$

In this expression, the c_k's, a_j's and b_j's are x^λ-functions such that the c_k's and the quotients $y_1^{\mu_j}/a_j$ and $b_j/y_1^{\mu_j}$ have values in $[-1,1]$. The functions V_n are analytic units. We apply the hypothesis of induction (on the integer s) to the family of functions of the variables $(\bar{y},z)$ constituted by the A_n's and the functions c_k, a_j and b_j. It comes:

$$A_n(\bar{y},z)=w_{A_n}(x)\prod_{i=1}^{t}\bar{y}_i(x)^{\delta_i^n}V_{A_n}(u_1(x),\ldots,u_p(x),m_1(x),\ldots,m_q(x)),$$

$$c_k(\bar{y},z)=w_{c_k}(x)\prod_{i=1}^{t}\bar{y}_i(x)^{\nu_i^k}V_{c_k}(u_1(x),\ldots,u_p(x),m_1(x),\ldots,m_q(x)),$$

$$a_j(\bar{y},z)=w_{a_j}(x)\prod_{i=1}^{t}\bar{y}_i(x)^{\zeta_i^j}V_{a_j}(u_1(x),\ldots,u_p(x),m_1(x),\ldots,m_q(x)),$$

$$b_j(\bar{y},z)=w_{b_j}(x)\prod_{i=1}^{t}\bar{y}_i(x)^{\xi_i^j}V_{b_j}(u_1(x),\ldots,u_p(x),m_1(x),\ldots,m_q(x)).$$

The functions $\bar{y}_i$ are $\mathfrak{L}\mathfrak{E}$-functions of the form $\bar{y}_i=\exp\bar{h}_i$ and we can assume one more time that there is no x^λ-comparability relation between y_1 and the $\bar{y}_i$'s.

Moreover, in the preceding expressions, the functions w_{A_n}, w_{c_k}, w_{a_j} and w_{b_j} are $\mathfrak{L}\mathfrak{X}$-functions and the analytic units have values in $[1/2,2]$. The functions c_k having values in $[-1,1]$, the functions $w_{c_k}\prod_{i=1}^{t}\bar{y}_i(x)^{\nu_i^k}$ are bounded (with values in $[-2,2]$). If we put

$$\tilde{c}_k=\frac{1}{2}w_{c_k}(x)\prod_{i=1}^{t}\bar{y}_i(x)^{\nu_i^k},$$

we can write the functions c_k under the following form:

$$c_k=\tilde{V}_{c_k}(\tilde{c}_k,u_1(x),\ldots,u_p(x),m_1(x),\ldots,m_q(x))$$

where the $\tilde{V}_{c_k}$'s are analytic units in a neighbourhood of $[-1,1]^{1+p+q}$. The functions $\tilde{c}_k$ have the same form as the functions u_i or m_i.

Moreover, we have:

$$\frac{y_1^{\mu_j}}{a_j(\bar{y},z)}=\tilde{w}_{a_j}(x)y_1^{\mu_j}\prod_{i=1}^{t}\bar{y}_i(x)^{-\zeta_i^j}V_{a_j}^{-1}(u_1(x),\ldots,u_p(x),m_1(x),\ldots,m_q(x))$$

where $\tilde{w}_{a_j}$ is the inverse of w_{a_j}. The function

$$\tilde{a}_j = \frac{1}{2}\tilde{w}_{a_j}(x)y_1^{\mu_j}\prod_{i=1}^{t}\bar{y}_i(x)^{-\zeta_i^j}$$

has values in $[-1, 1]$ and, as before, it comes:

$$\frac{y_1^{\mu_j}}{a_j(\bar{y}, z)} = \tilde{V}_{a_j}(\tilde{a}_j, u_1(x), ..., u_p(x), m_1(x), ..., m_q(x))$$

where the $\tilde{V}_{a_j}$'s are analytic units in a neighbourhood of $[-1, 1]^{1+p+q}$. A similar argument allows us to write the quotients $b_j/y_1^{\mu_j}$ under the form:

$$\frac{b_j(\bar{y}, z)}{y_1^{\mu_j}} = \tilde{V}_{b_j}(\tilde{b}_j, u_1(x), ..., u_p(x), m_1(x), ..., m_q(x)).$$

If we rename $u_{p+1}, ..., u_{p+p'}$ or $m_{q+1}, ..., m_{q+q'}$ the functions $\tilde{c}_k$, $\tilde{a}_j$ and $\tilde{b}_j$, we get:

$$\begin{aligned} f_n(x) = & w_{A_n}(x)y_1(x)^{\mu_0^n}\prod_{i=1}^{t}\bar{y}_i(x)^{\delta_i^n}\tilde{V}_n(u_1(x), ... \\ & ..., u_{p+p'}(x), m_1(x), ..., m_{q+q'}(x)). \end{aligned}$$

This ends the induction on s and show that $\mathbf{P}_e$ is true. □

From Theorem 2.4 we can derive the following corollary. It will be used in the last section.

Corollary 3.1. *Let $g(x) = V(z_1(x), ..., z_p(x), m_1(x), ..., m_q(x))$ be a reduced $\mathfrak{LE}$-function on an interval $I =]0, \varepsilon[$. We have $g = g_1 + g_2$ where g_1 is a $\mathfrak{LX}$-function and $g_2 = \sum_{n\in\mathbf{N}} h_n$ where the sum $\sum_{n\in\mathbf{N}} |h_n|$ uniformly converges on I and the h_n's are $\mathfrak{LE}$-functions of the following form:*

$$h_n(x) = a_n(x)y_1(x)^{\alpha_1^n}...y_s(x)^{\alpha_s^n}.$$

In such an expression, the a_n's are $\mathfrak{LX}$-functions and $(\alpha_1^n, ..., \alpha_s^n) \in \mathbf{R}^s \setminus \{0\}$. Moreover, for all $n \in \mathbf{N}$, we have $h_{n+1}(x) = o(h_n(x))$ as $x \to 0$. In particular, the set of the functions h_n is totally ordered and has a greatest element h_0 at 0.

Proof. Let us write the function V under the form

$$V(X_1, ..., X_p, Y_1, ..., Y_q) = \sum_{I=(i_1,...,i_p,j_1,...,j_q)\in\mathbf{N}^{p+q}} a_I X_1^{i_1}...X_p^{i_p} Y_1^{j_1}...Y_q^{j_q}.$$

The components functions m_k of V are of the form:

$$m_k(x) = y_1(x)^{\gamma_1^k}...y_s(x)^{\gamma_s^k} c_k(x).$$

We define the sets $E = \{I \in \mathbf{N}^{p+q} \mid \forall k = 1...q,\ \sum_{\ell=1}^{s} j_\ell \gamma_\ell^k = 0\}$ and $F = \mathbf{N}^{p+q} \setminus E$, and the corresponding analytic functions

$$V_1(X, Y) = \sum_{I\in E} a_I X^i Y^j$$
$$V_2(X, Y) = \sum_{I\in F} a_I X^i Y^j.$$

The functions g_1 and g_2 are then given by

$$g_1(x) = V_1(z_1(x), ..., m_q(x))$$
$$g_2(x) = V_2(z_1(x), ..., m_q(x)).$$

The function g_1 is a $\mathfrak{L}\mathfrak{X}$-function and we can write $g_2 = \sum_n t_n$ where the sum $\sum_{n\in\mathbf{N}} |t_n|$ is uniformly convergent on I and the t_n's are of the form:

$$t_n(x) = b_n(x) y_1(x)^{\alpha_1^n}...y_s(x)^{\alpha_s^n}.$$

In this expression, the b_n's are $\mathfrak{L}\mathfrak{X}$-functions and $(\alpha_1^n, ..., \alpha_s^n) \in \mathbf{R}^s \setminus \{0\}$. As $\sum_n |t_n|$ uniformly converges, we can group by packs the t_n's for which the exponents $(\alpha_1^n, ..., \alpha_s^n)$ are the same. It comes:

$$g_2(x) = \sum_{n\in\mathbf{N}} h_n(x)$$

where

$$h_n(x) = a_n(x) y_1(x)^{\alpha_1^n}...y_s(x)^{\alpha_s^n}.$$

The functions a_n are $\mathfrak{L}\mathfrak{X}$-functions and, if $n \neq m$ then $(\alpha_1^n, ..., \alpha_s^n) \neq (\alpha_1^m, ..., \alpha_s^m)$. Moreover we have

$$\lim_{x\to 0} h_n(x) = 0.$$

Indeed, these functions are bounded and if we assume that $h_n(x) \to h \neq 0$ as $x \to 0$, we deduce that h_n is comparable with the constant function h. Hence there exists a $\mathfrak{LE}$-function θ, having value in a compact sub-interval of $]0, +\infty[$, such that, for all $x \in I$:

$$a_n(x)y_1(x)^{\alpha_1^n}...y_s(x)^{\alpha_s^n} = h\theta(x).$$

As $(\alpha_1^n, ..., \alpha_s^n) \in \mathbf{R}^s \setminus \{0\}$, we deduce that one of the functions y_i is x^λ-comparable with the functions y_k, $k \neq i$ and a_n which gives a contradiction. Now we have

$$\frac{h_n(x)}{h_m(x)} = a(x)y_1(x)^{\beta_1}...y_s(x)^{\beta_s}$$

where $(\beta_1, ..., \beta_s) \neq (0, ..., 0)$. Thus, by a similar argument, the limit at zero of the quotient h_n/h_m is infinite or equal to zero. This proves that the h_n's are totally ordered. The fact that there exists a greatest element among them is obvious. □

4 Consequences

The first consequence of the preparation theorem for $\mathfrak{LE}$-functions deals with their asymptotic expansions in the scale $\mathfrak{E}_{\mathbf{R}}$. Theorem 2.5 follows directly from Corollary 3.1 of the previous section, that is why we omit its proof.

The second consequence deals with the integrals of x^λ-functions on the fibers of a x^λ-function. From [So], we know that for almost all real exponents occuring in the definition of a x^λ-function, its integral on the fibers of a x^λ-function belongs to the class of $\mathfrak{LX}$-functions. In order to prove Proposition 2.1, it suffices to produce a x^λ-function f of $\mathbf{R}^2$ such that its integral admits a *divergent* asymptotic expansion in the scale $\mathfrak{E}_{\mathbf{R}}$. We argue as follows.

Proof. We fix an analytic function of two variables: $V(X, Y) = \sum_{i,j} a_{i,j} X^i Y^j$ where we assume that the coefficients $a_{i,j}$ are positive and such that $\sum_{i,j} a_{i,j} 2^{i+j}$ converges. In particular, $(1, 1)$ is a polyradius of convergence of V. For all $\gamma \in \mathbf{R}_+^*$, we define

$$f_\gamma(x, y) = V\left(y, \frac{x}{y^\gamma}\right)$$

if (x, y) belongs to the x^λ-cylinder

$$C_\gamma = \left\{(x, y) \in]0, 1[^2 |\ y > x^{\frac{1}{\gamma}}\right\},$$

and $f_\gamma(x, y) = 0$ if not. We denote Ψ_γ the map $(x, y) \mapsto (y, x/y^\gamma)$. Then $f_\gamma = V \circ \Psi_\gamma$ when restricted to the cylinder C_γ. Under this form, the function

f_γ is a bounded and reduced x^λ-function on the x^λ-cylinder C_γ. Its expression is the following:

$$f_\gamma(x, y) = \sum_{i,j} a_{i,j} x^j y^{i-\gamma j}. \tag{1}$$

Let us now choose $\gamma > 0$ and *irrational*. To get the function F_γ, we integrate f_γ on the cylinder

$$C_{2\gamma} = \left\{(x, y) \in]0, 1[^2 | \ y > x^{\frac{1}{2\gamma}}\right\}.$$

As $\frac{1}{2\gamma} < \frac{1}{\gamma}$, we have $C_{2\gamma} \subset C_\gamma$. We can then denote

$$F_\gamma(x) = \int_{x^{\frac{1}{2\gamma}}}^{1} f_\gamma(x, y) dy.$$

The function F_γ is bounded on $\mathbf{R}$ and equal to zero out of $]0, 1[$. It is then not difficult to get the following expression:

$$F_\gamma(x) = \sum_{i,j} a_{i,j} x^{\frac{j}{2}} \frac{x^{\frac{j}{2}} - x^{\frac{1+i}{2\gamma}}}{1 + i - \gamma j}. \tag{2}$$

Under this form, we clearly see that if γ is very well approximated by the rational numbers, then the expansion of F_γ in the scale $\mathfrak{S}_{\mathbf{R}}$ will be divergent. As it is always possible to find such an irrational γ, the proof is completed. □

Remark. In the above proof, we make explicit the appearence of so-called *small divisors*. It appears that the expression of the integrated function F_γ can be given in term of compensators of “Ecalle-Roussarie”. Recall that such a compensator is a function of the form $\mathfrak{c}_{\alpha,\beta}(x) = (x^\alpha - x^\beta)/(\alpha - \beta)$ where α and β are non negative. Then it comes:

$$F_\gamma(x) = -\frac{1}{2\gamma} \sum_{i,j} a_{i,j} x^{\frac{j}{2}} \mathfrak{c}_{\frac{j}{2}, \frac{1+i}{2\gamma}}(x). \tag{3}$$

5 On a conjecture of van den Dries and Miller

In this last section, we give a partial answer to the following conjecture of van den Dries and Miller [DM].

Conjecture. *There is no o-minimal structure lying strictly between* $\mathfrak{S}(\mathbf{R}_{an}^{\mathbf{R}})$ *and* $\mathfrak{S}(\mathbf{R}_{an,\exp})$.

From Theorem 2.4 and its consequences, we can prove the following:

Proposition 5.1. *Let* $\mathfrak{A}$ *be an o-minimal structure such that* $\mathfrak{S}(\mathbf{R}_{an}^{\mathbf{R}}) \subset \mathfrak{A} \subsetneq \mathfrak{S}(\mathbf{R}_{an,\exp})$. *Then any one variable function definable in* $\mathfrak{A}$ *is definable in* $\mathfrak{S}(\mathbf{R}_{an}^{\mathbf{R}})$.

In order to prove the conjecture, one would need a multidimensional version of Proposition 5.1. Unfortunately, the multidimensional version of Theorem 2.4 is not so accurate.

Let us begin with the proof of Corollary 2.1.

Proof. From Theorem 2.3, we can assume that f is reduced on $]0,1[$: for all $x \in]0,1[$,

$$f(x) = a x_0^{\alpha_0} \ldots x_r^{\alpha_r} V(m_1(x), \ldots, m_p(x)).$$

In such an expression, $a \in \mathbf{R}$, the α_i's are real and

$$m_j(x) = x_0^{\alpha_0^j} \ldots x_r^{\alpha_r^j}$$

where the α_i^j's are real exponents. Moreover we have $x_0 = |x - \theta_0|$, $x_1 = |\log x_0 - \theta_1|$,..., $x_r = |\log x_{r-1} - \theta_r|$, the θ_i's being real constants. As θ_0 is comparable with x on $]0,1[$, it comes $\theta_0 = 0$. By induction, we easily deduce that all the θ_i's are equal to 0. Hence $x_i = |\ell_i(x)|$ for all i. As the m_j's have values in $[-1,1]$, we have necessarily $\alpha_0^j \geq 0$ for all j. Let us put $m(x) = x_0^{\alpha_0} \ldots x_r^{\alpha_r}$. The functions m and m_j are functions of the scale $\mathfrak{E}_{\mathbf{R}}$ and we have:

$$f(x) = a m(x). \sum_{I=(i_1,\ldots,i_p)} a_I m_1(x)^{i_1} \ldots m_p(x)^{i_p}.$$

Thus we can write

$$f(x) = \sum_{n \in \mathbf{N}} a_n m_n(x)$$

where the sum $\sum_{n\in\mathbf{N}} |a_n m_n(x)|$ is uniformly convergent and the m_n's are functions of $\mathfrak{E}_{\mathbf{R}}$ and are ordered: for all $n \in \mathbf{N}$,

$$\lim_{x \to 0_+} \frac{m_{n+1}(x)}{m_n(x)} = 0.$$

Now, the exponent of x in any monomial m_n is of the form:

$$\alpha_0 + i_1 \alpha_0^1 + \ldots + i_p \alpha_0^p$$

where $i_k \in \mathbf{N}$ and the α_0^k's are non negative. Thus these exponents can accumulate only at $+\infty$. □

The proof of Proposition 5.1 is then the following.

Proof. Let $\mathfrak{A}$ be an o-minimal structure such that $\mathfrak{S}(\mathbf{R}_{an}^{\mathbf{R}}) \subset \mathfrak{A} \subsetneq \mathfrak{S}(\mathbf{R}_{an,\exp})$. From the Theorem of Miller [Mi2], we know that, either $\mathfrak{A}$ is polynomially bounded, or it contains the graph of the exponential function exp. As $\mathfrak{S}(\mathbf{R}_{an,\exp})$ is the smallest o-minimal structure containing the globally subanalytic sets and the graph of the exponential, it follows that $\mathfrak{A}$ is necessarily polynomially bounded.

Consider a $\mathfrak{A}$-function $f : \mathbf{R} \to \mathbf{R}$ and assume that it is bounded on $]0, \varepsilon[$ for sufficiently small $\varepsilon > 0$ (this can always be assumed up to an inversion) and not identically equal to zero (ultimately at zero). Assume that, for all $0 < \varepsilon' \leq \varepsilon$, the function $f_{|]0,\varepsilon'[}$ is not a x^λ-function (i.e. f is not a *germ* of x^λ-function near zero). We will now derive a contradiction.

As $\mathfrak{A}$ is a substructure of $\mathfrak{S}(\mathbf{R}_{an,\exp})$, f is also a $\mathfrak{LE}$-function. Up to a reduction of ε, we can prepare f on $]0, \varepsilon[$ to the following form:

$$f(x) = z(x).\prod_{i=1}^{s} y_i(x)^{\delta_i}.V(z_1(x), ..., z_p(x), m_1(x), ..., m_q(x)).$$

From [Mi2], either f is identically equal to zero on $]0, \varepsilon[$, or there exist $c_0, \lambda_0 \in \mathbf{R}$, $c_0 \neq 0$, such that $f(x) = c_0 x^{\lambda_0} + o(x^{\lambda_0})$ as $x \to 0$. As f is bounded near zero, we have $\lambda_0 \geq 0$. From the reduced expression of f, we get:

$$z(x).\prod_{i=1}^{s} y_i(x)^{\delta_i} = c_0 x^{\lambda_0} + o(x^{\lambda_0}).$$

And thus

$$\frac{1}{x^{\lambda_0}}.z(x).\prod_{i=1}^{s} y_i(x)^{\delta_i} = c_0 + o(1).$$

In this equality, $c_0 + o(1)$ is a $\mathfrak{LE}$-function which has value in a compact subset of $]0, +\infty[$ on $]0, \varepsilon[$ (one more time up to a reduction of ε). If one of the δ_i's is not equal to zero, we can express the corresponding y_i has the product of a x^λ-function in the other variables by $c_0 + o(1)$. This implies that y_i is x^λ-comparable with a x^λ-function in the other variables and this is a contradiction. Hence $\delta_i = 0$ for all i and we can apply Corollary 3.1 to get the following expression for f:

$$f = g + \sum_{n \in \mathbf{N}} m_n$$

where g is a bounded $\mathfrak{LX}$-function and the m_n's are $\mathfrak{LE}$-functions satisfying the conclusion of the corollary.

From the identity $f(x) - c_0 x^{\lambda_0} = o(x^{\lambda_0})$, $\lambda_0 \geq 0$, we can apply again the Theorem of Miller for the $\mathfrak{A}$-function $x \mapsto o(x^{\lambda_0})$. If this function is not ultimately identically equal to zero, it comes:

$$f_1(x) = f(x) - c_0 x^{\lambda_0} = c_1 x^{\lambda_1} + o(x^{\lambda_1})$$

where $c_1 \neq 0$ and $\lambda_1 > \lambda_0$. If we continue this process inductively, we can get 3 different cases.

a. After a finite number of steps, we get a function which is identically zero on $]0, \varepsilon'[$, $\varepsilon' > 0$.

b. We get an increasing sequence of positive real numbers $(\lambda_i)_{i \in \mathbf{N}}$ such that $\lambda_i \to +\infty$ as $i \to +\infty$.

c. We get an increasing sequence of positive real numbers $(\lambda_i)_{i \in \mathbf{N}}$ such that $\lambda_i \to \lambda$ as $i \to +\infty$ with $\lambda \geq 0$.

Case a. In this case, we have $f(x) = \sum_{i=0}^{k} c_i x^{\lambda_i}$ on a interval $]0, \varepsilon'[$. Hence f is x^{λ}-function near zero and this is a contradiction.

Case b. Assume that there exists $i_0 \in \mathbf{N}$ such that $x^{\lambda_{i_0}} = o(m_0(x))$ as $x \to 0$. If we make an expansion of f at the "order" m_0, it comes:

$$\begin{aligned} f(x) &= \sum_{i<i_0} c_i x^{\lambda_i} + o(m_0(x)) \\ &= g_0(x) + m_0(x) + o(m_0(x)) \end{aligned}$$

where g_0 is the function g truncated at the order m_0. This is still a $\mathfrak{L}\mathfrak{X}$-function and the preceding equalities impliy that m_0 is a $\mathfrak{L}\mathfrak{X}$-function too, which is a contradicion. Thus, necessarily the function m_0 (and also the m_n's, $n > 0$) is smaller than any positive power of x at zero: for all i, $m_0(x) = o(x^{\lambda_i})$. Identifying the two preceding expansions at all order x^{λ_i} as $i \to +\infty$, we deduce that $g(x) = \sum_{i \in \mathbf{N}} c_i x^{\lambda_i}$. As g is a $\mathfrak{L}\mathfrak{X}$-function and no function log appear in its expansion, g is in fact a x^{λ}-function. Consequently, $f - g$ belongs to $\mathfrak{A}$ and:

$$(f - g)(x) = m_0(x) + o(m_0(x)).$$

If m_0 is not ultimately identically equal to zero, this equation implies that $f - g$ is not equivalent to a positive power of x at the origin and this contradicts the Growth dichotomy Theorem of Miller. Hence m_0 is equal to zero on a certain $]0, \eta[$, $\eta > 0$, and $f = g$ on this interval which gives one more time a contradiction.

Case c. We use Corollary 2.1 to show that this case is not possible. Like in the preceding case, if there exists i_0 such that $x^{\lambda_{i_0}} = o(m_0(x))$ we get a contradiction. Hence we have: $m_0(x) = o(x^{\lambda_i})$ for all i as $x \to 0$. As the powers of x in the expansion of g do not accumulate at λ, there exists $\eta > 0$ such that, if $x^{\alpha_0}\ell_1(x)^{\alpha_1}...\ell_p(x)^{\alpha_p}$ and $x^{\beta_0}\ell_1(x)^{\alpha_1}...\ell_p(x)^{\beta_p}$ are two monomials of g, then we have

$$|\beta_0 - \alpha_0| > \eta. \qquad (**)$$

Let us choose i_0 such that $|\lambda_{i_0+1} - \lambda_{i_0}| < \eta/2$. If we truncate the expansion of f at the order λ_{i_0+1}, we get:

$$\begin{aligned} f(x) &= \sum_{i=0}^{i_0+1} c_i x^{\lambda_i} + o(x^{\lambda_{i_0+1}}) \\ &= \bar{g}(x) + o(x^{\lambda_{i_0+1}}) \end{aligned}$$

where $\bar{g}$ is the function g truncated at the order λ_{i_0+1}. This is a $\mathfrak{L}\mathfrak{X}$-function. The preceding equality implies that there are two monomials in the expansion of $\bar{g}$ such that $(**)$ is not satisfied. This contradiction completes the proof. $\square$

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